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$\mathcal{G}^{+++}$-invariant formulation of gravity and M-theories: exact intersecting brane solutions

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Abstract: The set of exact solutions of the non-linear realisations of the $\mathcal{G}^{+++}$ Kac-Moody algebras is further analysed. Intersection rules for extremal branes translate into orthogonality conditions on the positive real roots characterising each brane. It is proven that all the intersecting extremal brane solutions of the maximally oxidised theories have their algebraic counterparts as exact solutions in the $\mathcal{G}^{+++}$-invariant theories. The proof is extended to include the intersecting extremal brane solutions of the exotic phases of the maximally oxidised theories.

Keywords: p-branes, String Duality, Space-Time Symmetries, Classical Theories of Gravity.

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Contents

1. Introduction

A maximally oxidised theory associated with a simple group $\mathcal{G}$ is a theory of gravity coupled to forms and dilatons defined at the highest possible space-time dimension D which upon dimensional reduction down to three is expressible in terms of a coset space $\mathcal{G}/\mathcal{H}$ where $\mathcal{H}$ is the maximally compact subgroup of $\mathcal{G}$. The maximally oxidised actions $S_{\mathcal{G}}$ corresponding to all the simple group $\mathcal{G}$ have been classified [1] and they comprise in particular pure gravity in D dimensions, the bosonic part of the low energy effective action of M-theory and the low energy effective action of the bosonic string. It has been conjectured that these actions, or some extension of them, possess the much larger very-extended Kac-Moody symmetry $\mathcal{G}^{+++}$. $\mathcal{G}^{+++}$ algebras are defined form the Dynkin diagrams obtained from those of $\mathcal{G}$ by adding three nodes [2]. One first adds the affine node, then a second node connected to it by a single line to define the overextended $\mathcal{G}^{++}$ algebras, then similarly a third one connected to the second to define the very-extended $\mathcal{G}^{+++}$ algebras. Such $\mathcal{G}^{+++}$ symmetries were first conjectured in the above mentioned particular case [3, 4] and the extension to all $\mathcal{G}^{+++}$ was proposed in [5]. In a different development, the study of the properties of cosmological solutions in the vicinity of a space-like singularity, known as cosmological billiards [6] revealed an overextended symmetry $\mathcal{G}^{++}$ for all $\mathcal{G}$ [7, 8].

Motivated by these developments and by the approach to E_8^{++} proposed in reference [9], we formulated in [10] an explicit non-linear realisation for all simple $\mathcal{G}^{+++}$. These actions $\mathcal{S}_{\mathcal{G}}$ are defined with no a priori reference to space-time, which is expected to be generated dynamically. These $\mathcal{G}^{+++}$-invariant actions are proposed as substitutes for the original field theoretic models of gravity, forms and dilatons and hopefully contain new degrees of freedom such as those encountered in string theories. They are formulated recursively from a level decomposition [9, 11, 12] with respect to a subalgebra A_{D-1} where D turns out to be the space-time dimension. The fields appearing in the actions live in a coset space $\mathcal{G}^{+++}/K^{+++}$ where the subalgebra K^{+++} is invariant under a 'temporal involution' [10] which is different from the often used Chevalley involution. The temporal involution preserves the Lorentz algebra $\mathrm{SO}(1, D-1)$ and as a consequence the actions $\mathcal{S}_{\mathcal{G}}$ are Lorentz invariant at each level. Exact solutions of these $\mathcal{G}^{+++}$-theories describing the algebraic properties of the extremal branes of the corresponding maximally oxidised

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theories have been obtained [10], and the existence of 'dualities' for all $\mathcal{G}^{+++}$ have been traced to their group-theoretical origin. The 'dualities' are described by Weyl reflections in $\mathcal{G}^{+++}$.

Here we extend these results. In section 2 we find exact solutions of $\mathcal{G}^{+++}$ corresponding to *all* the extremal intersecting brane solutions of the maximally oxidised theories. To establish this result, we first show that the extremal branes and the intersection rules [13] characterising their intersections are neatly encoded in the $\mathcal{G}^{+++}$ algebras. Namely each extremal brane corresponds to a real positive root of the $\mathcal{G}^{+++}$ algebra and the intersection rules for the branes translate into orthogonality conditions on their roots. These results permit a truncation of the actions $\mathcal{S}_{\mathcal{G}}$ to their quadratic expansion from which the exact solutions are derived. In section 3 these results are further extended for all $\mathcal{G}^{+++}$ to 'exotic' phases, which may have different space-time signature. It has indeed been shown in [14] that the 'temporal' involution is not invariant under all the Weyl reflections. In the particular case of E_8^{+++} this implies that the exotic phases of M-theory [15, 16] reached by timelike T-duality in the string language are included in the formalism of [10]. In general exotic phases for all $\mathcal{G}^{+++}$ are also included.

2. Intersecting brane configurations as exact solutions in $\mathcal{G}^{+++}$

In [10], theories invariant under $\mathcal{G}^{+++}$ where constructed and exact solutions describing the algebraic properties of the BPS extremal brane solutions of all the maximally oxidised theories associated with the simple groups $\mathcal{G}$ were presented. In this section, we extend these results. We proof that all the intersecting extremal brane solutions of these theories have also their algebraic counterpart as exact solutions in $\mathcal{G}^{+++}$. As an element of this general proof we find the elegant encoding in the $\mathcal{G}^{+++}$ algebras of the intersections.

In order to establish this result we first analyse the relations between brane dynamics and symmetry in the maximally oxidised theories. We first recall the generic intersection rules [13] which determine how extremal branes can intersect orthogonally with zero binding energy. These intersection rules are valid for a generic theory in D dimensions which includes gravity, a dilaton, form field strengths F_{n_I} of arbitrary degree n_I and arbitrary couplings to the dilaton a_I.[1] They give for each pair (A, B) of q-branes of dimensions (q_A, q_B), the number of dimensions $\bar{q}$ on which they intersect in terms of the total number of space-time dimensions D and of the field strength couplings to the dilaton. They read [13]

$$\bar{q} + 1 = \frac{(q_A + 1)(q_B + 1)}{D - 2} - \frac{1}{2}\varepsilon_A a_A \varepsilon_B a_B \,, \tag{2.1}$$

where $\bar{q}$ is the number of spatial dimensions on which the q_A and the q_B brane intersect, ϵ_A is $+1$ (resp. -1) if the q_A-brane is electric (resp. magnetic).[2] An intersecting extremal brane configuration exists thus between two branes if $\bar{q}$ is an integer not bigger than the brane

[1] We consider $n_I \leq D/2$. If $n_I > D/2$, we can indeed replace the field strength we start with by its Hodge dual.

[2] The case $\bar{q} = -1$ is also relevant and have an interpretation in terms of instanton in the euclidean. Then ,the time coordinate doesn't need to be longitudinal to all the branes.

of lowest dimension. We restrict ourself to intersecting brane configurations characterised by a space which is asymptotically flat, namely we consider configurations with an overall transverse space $d \geq 3$. In the derivation of eq. (2.1), it is assumed that in the configuration considered there are no contributions to the equations of motion from the Chern-Simons terms that can be present in the action.

We are interested in intersecting brane configurations of the maximally oxidised theories corresponding to any simple group $\mathcal{G}$ [1] and characterised by at most one dilaton.[3] In ref. [17] it has been found that the scalar products between first electric and magnetic roots encountered in the dimensional reduction process and corresponding to any of the form field strengths are given by the intersection rules. More precisely one gets the following relations. First one has [17]

$$\begin{aligned} \alpha^e_A \cdot \alpha^e_B &= \bar{q}^{(e_A, m_B)} + 1 && (2.2)\\ &= q^e_A - \bar{q}^{(e_A, e_B)}\,, && (2.3) \end{aligned}$$

where A (resp. B) refers to the form field strength F_{n_A} (resp. F_{n_B}) present in the maximally oxidised $\mathcal{G}$ theory, and $n_A \leq n_B$. The α^e_X with $X = (A, B)$ is the first electric root coming from the F_{n_X} upon dimensional reduction, that is the one appearing when one reaches $n_X - 1$ compact dimensions. The superscripts of $\bar{q}$ label the electric or magnetic nature of the q_A-brane and the q_B-brane. One has $q^e_X = n_X - 2$ and $q^m_X = D - q^e_X - 4$. Using these relations, we express in eq. (2.3) the scalar product of the two first electric roots in terms of the intersection between the two corresponding electric branes. Second, one gets for the scalar products between a first electric root and a first magnetic root

$$\begin{aligned} \alpha^e_A \cdot \alpha^m_B &= \bar{q}^{(e_A, e_B)} + 1 && (2.4)\\ &= q^e_A - \bar{q}^{(e_A, m_B)}\,, && (2.5) \end{aligned}$$

with again $n_A \leq n_B$. The first magnetic root α^m_X corresponds in the dimensional reduction to the additional scalar coming from F_{n_X} which arises when one reaches $n_X + 1$ non-compact dimensions and is obtained by dualizing the n_X-form. We have expressed the scalar product in terms of the intersection of q^e_A and q^m_B (note that since in our setting $n_X \leq D/2$, one has always $q^e_X \leq q^m_Y$). Third, the scalar product between two first magnetic roots is given by

$$\begin{aligned} \alpha^m_A \cdot \alpha^m_B &= \bar{q}^{(e_A, m_B)} + 1 && (2.6)\\ &= q^m_B - \bar{q}^{(m_A, m_B)}\,. && (2.7) \end{aligned}$$

Here $q^m_B \leq q^m_A$. Thus the smallest brane appears always in the r.h.s. of eqs. (2.3), (2.5) and (2.7).

We now use these results in the context of the actions proposed in [10] which are explicitly invariant under the very-extended algebras $\mathcal{G}^{+++}$.

We recall how the $\mathcal{G}^{+++}$-invariant actions $\mathcal{S}_\mathcal{G}$ were constructed recursively from a level expansion with respect to a subalgebra A_{D-1} where D is the space-time dimension [10]. At

[3] We are considering all the maximally oxidised theory except the ones corresponding to the C_{q+1}-series. The maximally oxidised theory C_{q+1} is a four dimensional theory which contains q-dilatons.

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each level the $SO(1, D-1)$ invariance is realised through the use of the 'temporal' involution instead of the usual Chevalley one in the construction of the non-linear realisation. $\mathcal{G}^{+++}$ contains a subalgebra $GL(D)$ and we have $SL(D) \subset GL(D) \subset \mathcal{G}^{+++}$. The generators of the $GL(D)$ subalgebra are taken to be $K^a{}_b$ $(a, b = 1, 2, \ldots, D)$ with commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b\,. \tag{2.8}$$

The $K^a{}_b$ along with the the abelian generator R (present when the corresponding $\mathcal{G}$ theory has one dilaton) are the level zero generators. The operators of level greater than zero are tensors of the A_{D-1} subalgebra. The lowest levels contain antisymmetric tensor step operators $R^{a_1 a_2 \ldots a_r}$ associated with the electric and magnetic roots occurring in the dimensional reduction of the corresponding maximally oxidised theory. They satisfy

$$[K^a{}_b, R^{a_1\ldots a_r}] = \delta^{a_1}_b R^{a a_2 \ldots a_r} + \cdots + \delta^{a_r}_b R^{a_1 \ldots a_{r-1} a}\,, \tag{2.9}$$

$$[R, R^{a_1\ldots a_r}] = -\frac{\varepsilon_A a_A}{2} R^{a_1\ldots a_r}\,, \tag{2.10}$$

where a_A is the dilaton coupling of the corresponding form field strength in the $\mathcal{G}$ theory and ε_A is $+1$ (-1) if the corresponding root is electric (magnetic). One defines fields in a one dimensional space ξ, a priori unrelated to space-time, as the parameters of the group elements $\mathcal{V}$ built out of Cartan and positive step operators in $\mathcal{G}^{+++}$. It takes the form

$$\mathcal{V} = \exp\Big(\sum_{a\geq b} h_b{}^a(\xi) K^b{}_a - \phi(\xi) R\Big) \exp\Big(\sum \frac{1}{r!s!} A_{b_1\ldots b_s}{}^{a_1\ldots a_r}(\xi) R^{b_1\ldots b_s}{}_{a_1\ldots a_r} + \cdots\Big), \tag{2.11}$$

where the first exponential contains only level zero operators and the second one the positive step operators of level strictly greater than zero. In terms of these fields, the action $\mathcal{S}_\mathcal{G}$ is

$$\mathcal{S}_\mathcal{G} = \mathcal{S}_\mathcal{G}{}^{(0)} + \sum_A \mathcal{S}_\mathcal{G}{}^{(A)}\,, \tag{2.12}$$

where $\mathcal{S}_\mathcal{G}{}^{(0)}$ contains all level zero contributions. Explicitly one has

$$\mathcal{S}_\mathcal{G}{}^{(0)} = \frac{1}{2}\int d\xi \frac{1}{n(\xi)}\left[\frac{1}{2}(g^{\mu\nu} g^{\sigma\tau} - \frac{1}{2} g^{\mu\sigma} g^{\nu\tau}) \frac{dg_{\mu\sigma}}{d\xi}\frac{dg_{\nu\tau}}{d\xi} + \frac{d\phi}{d\xi}\frac{d\phi}{d\xi}\right] \tag{2.13}$$

$$\mathcal{S}_\mathcal{G}{}^{(A)} = \frac{1}{2}\int d\xi \frac{1}{n(\xi)}\left[\frac{1}{r!s!}\exp(-2\lambda\phi)\frac{DA_{\mu_1\ldots\mu_r}{}^{\nu_1\ldots\nu_s}}{d\xi} g^{\mu_1\mu'_1}\cdots g^{\mu_r\mu'_r} g_{\nu_1\nu'_1}\cdots g_{\nu_s\nu'_s}\frac{DA_{\mu'_1\ldots\mu'_r}{}^{\nu'_1\ldots\nu'_s}}{d\xi}\right]$$

Here, $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$, $e_\mu{}^a = (e^{-h(\xi)})_\mu{}^a$ with (a, b) Lorentz indices and (μ, ν) $GL(D)$ indices, λ is the generalization of $-\varepsilon_A a_A/2$ to all roots and $D/D\xi$ the non-linear covariant derivative generalising $d/d\xi$ to take into account non vanishing commutators between positive level step operators. The arbitrary lapse function $n(\xi)$ renders $\mathcal{S}_\mathcal{G}$ reparametrisation invariant.

The exact solution of the $\mathcal{G}^{+++}$ theories [10] corresponding to a single extremal q-brane longitudinal to the $\lambda_1 \ldots \lambda_q$ spatial directions is always electrically described[4] and is

[4] If the brane is a magnetic one, it is described by the Hodge dual field whose potential appear as well in the low levels. The time direction is taken to be the direction $\hat{1}$ thus $x^1 = t$.

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characterised by only one non-zero field component $A_{t\lambda_1\ldots\lambda_q}$. This field is the parameter of an antisymmetric tensor step operator of low level $R^{1\lambda_1\cdots\lambda_q}$ and we denote the corresponding *real positive root*[5] by $\alpha_{(1,\lambda_1,\ldots,\lambda_q)}$.

Suppose now that we have two branes, one extremal q_A-brane along the $\lambda_1\ldots\lambda_{q_A}$ spatial directions associated with the root $\alpha_{(1,\lambda_1,\ldots,\lambda_{q_A})}$ and an extremal q_B-brane with $q_B \geq q_A$ along the $\nu_1\ldots\nu_{q_B}$ spatial directions associated with the root $\alpha_{(1,\nu_1,\ldots,\nu_{q_B})}$. We assume that the branes have $\bar{q}$ indices in common namely $\lambda_i = \nu_i$ for $\bar{q}$ different i's. We want first to demonstrate the following statement which translates in a group language the intersection rules

Theorem 1. *The existence of an integer solution $\bar{q} \leq q_A$ of the intersection rule equation between two extremal branes eq. (2.1) is equivalent to the following condition*[6] *on the two real positive roots corresponding to the two branes*

$$\alpha_{(1,\lambda_1,\ldots,\lambda_{q_A})} \cdot \alpha_{(1,\nu_1,\ldots,\nu_{q_B})} = 0\,. \tag{2.14}$$

In order to show that we will use the relations between first electric and magnetic roots derived in the context of the dimensional reduction. We first note that the scalar product between two roots is invariant under Weyl reflection and that the Weyl reflections generated by the simple roots α_i^g, $i = 1,\ldots,D-1$ of the gravity line defined by the subalgebra A_{D-1} simply exchanges $x^i \leftrightarrow x^{i+1}$ (including time $t = x^1$). Using Weyl reflections, we can thus map $\alpha_{(1,\lambda_1,\ldots,\lambda_{q_A})}$ onto $\alpha_{(D-q_A,D-q_A+1,\ldots,D)}$ and map $\alpha_{(1,\nu_1,\ldots,\nu_{q_B})}$ onto $\alpha_{(D-q_A-q_B+\bar{q},\ldots,D-q_A-1,D-\bar{q},\ldots,D)}$. These roots share the directions $D-\bar{q},\ldots,D$. We have

$$\begin{aligned}\alpha_{(1,\lambda_1,\ldots,\lambda_{q_A})} \cdot \alpha_{(1,\nu_1,\ldots,\nu_{q_B})} &= \alpha_{(D-q_A,D-q_A+1,\ldots,D)} \cdot \alpha_{(D-q_A-q_B+\bar{q},\ldots,D-q_A-1,D-\bar{q},\ldots,D)} \\ &\equiv \alpha_A \cdot \alpha'_B\,.\end{aligned} \tag{2.15}$$

Since we made the assumption that in the intersecting brane configurations the overall transverse space is $d \geq 3$, we have $D - q_A - q_B + \bar{q} \geq 4$ and $\{\alpha_A, \alpha'_B\} \in \mathcal{G} \subset \mathcal{G}^{+++}$. The root α_A corresponds to the generator $R^{D-q_A\ldots D}$ and, in the dimensional reduction language is a first electric (or magnetic) root. The root α'_B corresponds to the generator $R^{D-q_A-q_B+\bar{q}\ldots D-q_A-1\ D-\bar{q}\ldots D}$ and it is not a first electric or magnetic root. In order to be able to use the results eqs. (2.3), (2.5) and (2.7) to compute $\alpha_A \cdot \alpha'_B$ we need to express the root α'_B in terms of the simple roots of the gravity line and in terms of the first root α_B corresponding to the generator $R^{D-q_B D-q_B-1\ldots D}$. We can relate $R^{D-q_A-q_B+\bar{q}\ldots D-q_A-1\ D-\bar{q}\ldots D}$ to $R^{D-q_B D-q_B-1\ldots D}$ with eq. (2.9). Using the fact that the simple roots α_i^g of the gravity line correspond to the generators $K^i{}_{i+1}$, one finds the following relations between α_B and α'_B

$$\alpha'_B = \alpha_B + (q_A - \bar{q})\alpha^g_{D-q_a-1} + \Lambda\,, \tag{2.16}$$

[5] In ref. [18], they argued in the context of $E_{10} = E_8^{++}$ that extremal branes correspond to some imaginary roots. It is worthwhile to emphasise that in the present $\mathcal{G}^{+++}$ framework *all* the extremal branes correspond to real positive roots.

[6] This condition has been noticed, in a somewhat different setting, for some particular intersecting configurations of M-theory in [19].

where Λ is a sum of simple roots of the gravity line with positive integer coefficients which *does not contain* a contribution from $\alpha^g_{D-q_a-1}$. We have thus

$$\Lambda \cdot \alpha_A = 0\,. \tag{2.17}$$

Now using eqs. (2.3), (2.5) and (2.7) together with eqs. (2.16) and (2.17), we find

$$\begin{aligned} \alpha_A \cdot \alpha'_B &= \alpha_A . \alpha_B + (q_A - \bar{q})\, \alpha_A \cdot \alpha^g_{D-q_A-1} \\ &= q_A - \bar{q} + (q_A - \bar{q})(-1) \\ &= 0\,, \end{aligned} \tag{2.18}$$

which demonstrates theorem 1.

We are now in position to construct exact solutions of the $\mathcal{G}^{+++}$ actions corresponding to extremal intersecting brane solutions. It will be sufficient for the construction to analyse pairwise intersections. Consider two branes, one corresponding to the root $\beta \equiv \alpha_{(1,\lambda_1,\ldots,\lambda_{q_A})}$ and the other to $\gamma \equiv \alpha_{(1,\nu_1,\ldots,\nu_{q_B})}$. The branes have $\bar{q}$ indices in common namely $\lambda_i = \nu_i$ for $\bar{q}$ different i's. We label the two roots such that the level of γ is not lower than the level of β. We also lift the restriction that the overall transverse space $d \geq 3$, namely we admit intersecting brane configurations which are no longer necessarily in a $\mathcal{G} \subset \mathcal{G}^{+++}$. We claim

Theorem 2. *There exists a solution of the $\mathcal{G}^{+++}$ action describing the intersection of two extremal branes associated with the real positive roots β and γ iff*

$$\beta \cdot \gamma = 0 \tag{2.19}$$

and

$$\beta + \gamma \neq \textit{root}\,. \tag{2.20}$$

To establish theorem 2 we will use the following lemma

Lemma 1. *If eq. (2.19) is satisfied then eq. (2.20) is equivalent to*

$$\gamma \neq \beta + \tilde{\alpha}\,, \tag{2.21}$$

where $\tilde{\alpha}$ is any positive root of $\mathcal{G}^{+++}$.

To proof the lemma we first show, using eq. (2.19) and eq. (2.20), that $\gamma - \beta$ is not a root namely that $[E_\gamma, F_\beta] = 0$ where F_β is the step operator corresponding to the negative root $-\beta$. This is a consequence of the Jacobi identity $[F_\beta[E_\gamma, E_\beta]] + [E_\gamma[E_\beta, F_\beta]] + [E_\beta[F_\beta, E_\gamma]] = 0$. The first term is zero because of eq. (2.20). To evaluate the second term we use the standard invariant bilinear form K defined on a Kac-Moody algebra [20]. We consider the following identity reflecting the invariance of K: $K([E_\alpha, F_\alpha], H_n) + K(F_\alpha, [E_\alpha, H_n]) = 0$ where H_n is the Cartan generator in the Chevalley basis corresponding to the simple root α_n. Writing α as a sum of simple roots α_m: $\alpha = \sum_{m=1}^{r} N_m \alpha_m$ where N_m are non-negative integers and using the fact that for $\mathcal{G}^{+++}$ the determinant of the Cartan matrix is different from zero, we deduce from the above identity that $[E_\alpha, F_\alpha] = (1/2) K(E_\alpha, F_\alpha) \sum_m N_m \alpha_m^2\, H_m$.

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From there it follows that $[E_\gamma, [E_\beta, F_\beta]] = -K(E_\beta, F_\beta)\,(\beta\cdot\gamma)\,E_\gamma$. Thus this term is also zero by eq. (2.19). Furthermore, since γ is a root different from β, $[E_\beta[F_\beta E_\gamma]] = 0$ only if $[F_\beta E_\gamma] = 0$ thus eq. (2.20) $\Rightarrow$ eq. (2.21). Using eq. (2.19) and the same Jacobi identity one shows also that eq. (2.21) $\Rightarrow$ eq. (2.20). In that case the second and the third term are zero and the first one implies that $\beta+\gamma$ is not a root. This concludes the proof of the lemma.

An exact solution of $\mathcal{S}_{\mathcal{G}}$ corresponding to an intersecting brane configuration is constructed as a generalisation of single extremal brane solutions discussed in [10]. For each of the $\mathcal{N}$ branes present in the configuration the solution has a non-zero field component given by

$$A_{t\lambda_1\ldots\lambda_{q_A}} = \epsilon_{t\lambda_1\ldots\lambda_{q_A}} \left[\frac{2(D-2)}{\Delta_A}\right]^{1/2} H_A^{-1}(\xi)\,, \qquad A = 1,\ldots,\mathcal{N}\,, \tag{2.22}$$

and, defining $p^{(a)} = p^{(\mu)} = \ln e^a_\mu$ for the diagonal vielbein in a triangular gauge, dilaton and metric components given by

$$p^{(\mu)} = \sum_{A=1}^{\mathcal{N}} p_A^{(\mu)} = \sum_{A=1}^{\mathcal{N}} \frac{\eta_A^\mu}{\Delta_A} \ln H_A(\xi) \tag{2.23}$$

$$\phi = \sum_{A=1}^{\mathcal{N}} \phi_A = \sum_{A=1}^{\mathcal{N}} \frac{D-2}{\Delta_A} \varepsilon_A a_A \ln H_A(\xi)\,. \tag{2.24}$$

Here $\eta_A^\mu = q_A + 1$ or $-(D - q_A - 3)$ depending on whether the direction $\hat{\mu}$ is perpendicular or parallel to the q_A-brane and $\Delta_A = (q_A+1)(D-q_A-3) + \frac{1}{2}a_A^2(D-2)$. Each of the branes in the configuration is characterised by one harmonic function in ξ-space, namely one has

$$\frac{d^2 H_A(\xi)}{d\xi^2} = 0 \qquad A = 1,\ldots,\mathcal{N}\,. \tag{2.25}$$

In order to show that eqs. (2.22)–(2.25) constitute indeed a solution of the equations of motion we proceed in two steps.

First we make the working hypothesis that we can substitute $\mathcal{S}_{\mathcal{G}}$ by its quadratic truncation, as we did for the single brane solution in reference [10]. The quadratic action is defined by expanding $\mathcal{S}_{\mathcal{G}}$ given in eq. (2.12) in power of fields parametrizing the positive step operators up to quadratic terms. Under this assumption, we check that eqs. (2.22)–(2.25) are solutions of the equations of motion (eqs. (3.10)–(3.14) of ref. [10]). One potential problem arises from the argument of the exponential in front of a $dA_{t\lambda_1\ldots\lambda_{q_A}}/d\xi$ term associated with a q_A-brane present in the configuration: this argument is given by $\varepsilon a\phi - 2p^{(t)} - 2\sum_{\lambda=\lambda_1}^{\lambda_r} p^{(\lambda)}$. We verify however that the contribution to this expression of any other q_B-brane present in the configuration vanishes identically. This is easily shown using the intersection rule eq. (2.1) between q_A and q_B, which is equivalent, as shown in theorem 1, to eq. (2.19). Note that it follows from the above solution and from the intersection rules that the lapse constraint is satisfied and takes the form

$$\sum_{\alpha=1}^{D} (dp^{(\alpha)})^2 - \frac{1}{2}\left(\sum_{\alpha=1}^{D} dp^{(\alpha)}\right)^2 + \frac{1}{2}(d\phi)^2 - \sum_{A=1}^{\mathcal{N}} \frac{D-2}{\Delta_A}(d\ln H_A)^2 = 0\,. \tag{2.26}$$

Provided the truncation of the $\mathcal{S}_{\mathcal{G}}$ actions to their quadratic form is consistent we have thus an exact solution. The conditions eq. (2.20) and eq. (2.21) are precisely those ensuring that the replacement of the actions by their quadratic simplified versions is consistent. First, eq. (2.20) ensures that the substitution of the solution eqs. (2.22)–(2.25) in the action $\mathcal{S}_{\mathcal{G}}$ leads only to quadratic terms. Second, we check that for the field components $\tilde{A}$ which are zero in the configuration, $\tilde{A} = 0$ is solution of the equations of motion of the full $\mathcal{S}_{\mathcal{G}}$. This is the case. Indeed, on the one hand, eq. (2.20) ensures that the covariant derivative of $\tilde{A}$ does not contain non-linear terms built purely out of non-zero field components in the configuration. On the other hand, eq. (2.21) ensures that in the covariant derivative of a non-zero A field component in the configuration there are no non-linear terms which would contain other non-zero A field components in the configuration along with a $\tilde{A}$ field component. The equations of motion of the $\tilde{A}$'s are thus trivially satisfied putting these $\tilde{A}$'s to zero. This concludes the proof of theorem 2.

In addition to the above considered charged extremal brane, there exist two other gravitational BPS branes in the $\mathcal{G}^{+++}$ theories, namely the KK-momentum and the KK-monopole. These are related by Weyl reflections to the charged BPS branes [10] and since Weyl reflections preserve the scalar product, theorem 2 extends to configurations containing also gravitational branes. The root of a KK-momentum in the x^k direction is the one associated with the positive level zero step operator $K^1{}_k$ and the root of the KK-monopole with, say, the longitudinal directions $(x^2, \ldots, x^{D-4})$ and Taub-NUT direction x^D is the one associated with the positive step operator $R^{1\ldots D-4D,D}$ (see [10, appendix B]). This step operator, which is antisymmetric in the first $D-3$ indices and with a vanishing totally antisymmetrised contribution, exists at some level for all $\mathcal{G}^{+++}$ [12].

We now turn to our central theorem

Theorem 3. *There is a one to one correspondence between the exact solutions of $\mathcal{S}_{\mathcal{G}}$ given by theorem 2 and the intersecting extremal brane solutions of the maximally oxidised theory $S_{\mathcal{G}}$.*

We first proof the theorem for $\mathcal{G}$ simply laced. First recall that the intersection rules eq. (2.1) characterising the intersecting brane solutions of the maximally oxidised theories have been derived under the assumption that in the given configuration the Chern-Simons terms do not contribute to the equations of motion. This is the case for all the phases of M-theory ($11D$ SUGRA, IIA and IIB) [13] . More generally one can check by inspection of the explicit form of the actions [1] that for all maximally oxidised theory corresponding to simply laced group $\mathcal{G}$, it is also the case. Second, we note that for simply laced $\mathcal{G}^{+++}$ theories theorem 2 simplifies. In that case, eq. (2.20) and thus also eq. (2.21) are trivially satisfied once the eq. (2.19) is implemented. Indeed, for simply laced theories, all the real roots α_i^R have the same length (say $(\alpha_i^R)^2 = 2$) thus if $\alpha_i^R \cdot \alpha_j^R = 0$, $\alpha_+ \equiv \alpha_i^R + \alpha_j^R$ can not be a root because $\alpha_+^2 = 4$ and eq. (2.20) is satisfied. Consequently, eq. (2.20) and eq. (2.21), eliminating potential problems related to the non-linear terms in the equations of motion of $\mathcal{S}_{\mathcal{G}}$, are trivially satisfied. Thus the intersection rule, which in $\mathcal{G}^{+++}$ is implemented by eq. (2.19), fixes uniquely the intersecting brane solutions, both in $S_{\mathcal{G}}$ and in $\mathcal{S}_{\mathcal{G}}$. This establish the proof of theorem 3 for simply laced $\mathcal{G}$.

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We now turn to the non-simply laced theories with one dilaton namely the B_{D-2} series and F_4 (G_2 does not admit intersecting brane solutions).

We discuss first in detail the B_{D-2} series. The maximally oxidised $\mathcal{G} = B_{D-2}$ theory in D-dimensions with one dilaton contains a three-form field strength F_3 with dilaton coupling $a_3 = -l$ and a two-form field strength F_2 with dilaton coupling $a_2 = -l/2$ where $l^2 = 8/(D-2)$. There is an electric extremal 0-brane and a magnetic $(D-4)$-brane associated with F_2 and an electric 1-brane and a magnetic $(D-5)$-brane associated with F_3. In ten dimension it corresponds to the bosonic part of the low energy effective action of the heterotic string truncated to only one gauge field. The dilaton couplings are such that the intersection rules eq. (2.1) predict among the possible intersecting brane configurations the following ones associated with F_2: $0 \cap 0 = -1$, $0 \cap (D-4) = 0$ and $(D-4) \cap (D-4) = D-5$. Inspecting the action of the maximally oxidised theory B_{D-2} given in [1], it is easy to see there is a Chern-Simons like term appearing in the Bianchi identity $dF_3 = (1/2)\, F_2 \wedge F_2$, which does not vanish in these three configurations. Consequently these intersections are not solutions of the maximally oxidised theory and do not exist. The other possible intersecting brane configurations predicted by eq. (2.1) are not invalidated by such Chern-Simons like term and do exist.

In the $\mathcal{G}^{+++} = B_{D-2}^{+++}$ theory these three configurations are also discarded because they violate eq. (2.20) of theorem 2 (or equivalently eq. (2.21)). To see that, we recall the level decomposition of B_{D-2}^{+++} in terms of A_{D-1} [12]. There are two simple roots which do not belong to the gravity line. The first one α_D is short and corresponds to R^D associated with the component A_D of the one-form potential. The second root not belonging to the gravity line α_{D+1} is a long one and corresponds to $R^{5\dots D}$ associated with the component $A_{5\dots D}$ of the $(D-4)$-form potential. The level decomposition in terms of A_{D-1} representations is thus labelled by two non-negative integers (l_1, l_2) giving the number of time the two roots (α_{D+1}, α_D) appear in a given representation. The lowest level corresponding to the different potentials associated with branes are the following [12]

level	α^2	potential
$(0,1)$	1	A_1
$(0,2)$	2	A_2
$(1,0)$	2	A_{D-4}
$(1,1)$	1	A_{D-3}

Now it is easy to see that the three configurations are eliminated. The configuration $0 \cap 0 = -1$ does not satisfy eq. (2.20) indeed the sum of the two roots corresponding to the two 0-branes is a root of level $(0,2)$. For the configuration $(D-4) \cap 0 = 0$ it is easier to see that eq. (2.21) is violated. Indeed the root of level $(1,1)$ corresponding to the $(D-4)$-brane is the sum of the root of level $(0,1)$ corresponding to the 0-brane and of a root of level $(1,0)$ corresponding to the potential $A_{y_1 \dots y_{D-4}}$ where the y_i are the spatial longitudinal coordinates of the $(D-4)$-brane. Finally, as far as the configuration $(D-4) \cap (D-4) = D-5$ is concerned, we need to know the level decomposition at higher levels. At level $(2,2)$ there is a representation [12] corresponding to real long roots characterised by the following Dynkin labels:[7] $(1,0,\ldots,0,1,0)$. The sum of the roots corresponding to the two $(D-4)$-brane is a long root belonging to that representation. Consequently this configuration is not solution.

We thus conclude that each time there is an intersecting brane solution in the maximally oxidised B_{D-2} theory, its algebraic counterpart exists as an exact solution of B_{D-2}^{+++} theory. The solutions predicted by the intersection rules eq. (2.1) which are eliminated because of Chern-Simons terms in the maximally oxidised theory are also eliminated in B_{D-2}^{+++} because of the existence of non-linear terms in the configuration. Furthermore the intersections between pair of branes which are discarded correspond always to configuration containing two branes, magnetic or electric, corresponding to two short roots.

A similar discussion can be performed in the F_4 case. The maximally oxidised theory $\mathcal{G} = F_4$ is a six dimensional theory with one dilaton, a one-form field strength, two two-form field strengths and two three-form field strengths [1]. There are a lot of possible intersecting brane pairs in this theory. It suffices to consider the configurations involving two extremal branes corresponding to two short roots. They are associated with several field strengths, the two two-form field strengths F_2 and F_2' with dilaton couplings $a_2 = -1/\sqrt{2}$ and $a_2' = 1/\sqrt{2}$, the one-form field strength F_1 with dilaton coupling $a_1 = \sqrt{2}$ and the self-dual three-form field strength F_3. To each two-form corresponds an electric 0-brane and a magnetic 2-brane, a (-1)-brane and a 3-brane are associated with F_1 and a self-dual 1-brane is associated with F_3. The intersection rules predict the following configurations between these electric and magnetic branes: $0 \cap 0 = -1$, $0 \cap 2 = 0$, $2 \cap 2 = 1$, $0' \cap 0' = -1$, $0' \cap 2' = 0$, $2' \cap 2' = 1$, $1 \cap 1 = 0$, $-1 \cap 1 = -1$, $3 \cap 1 = 1$. These intersections predicted by eq. (2.1) are not solutions of the maximally oxidised theory. Indeed as for the B_{D-2}, there are Chern-Simons type terms (see [1]) which are non zero in these configurations. The other intersections predicted by eq. (2.1) are solutions. Again, these configurations are also eliminated in the $\mathcal{G}^{+++} = F_4^{+++}$ theory. Using the level decomposition of F_4^{+++} (see table 9 and A6 of ref. [12]), one can show that in these configurations eq. (2.20) and eq. (2.21) are not satisfied. We have thus again a perfect agreement between the existence of intersecting extremal brane solutions in the maximally oxidised F_4 theory and the existence of the algebraic counterpart in F_4^{+++}. This concludes the proof of our central theorem 3.

3. Extension to the exotic phases

In superstring theories, the U-duality group corresponds to the Weyl group of E_8^{+++} [21, 22, 23, 5]. In the type IIA language the non-trivial Weyl reflection generated by the simple root which do not belong to the gravity line of E_8^{+++} corresponds to a double T-duality in the $\hat{9}$ and $\widehat{10}$ directions plus an exchange of these two directions. Combining this Weyl reflection with the ones of the gravity line one is inevitably led to consider T-duality involving the timelike direction. Compactification of the timelike direction in string theories [15] along with timelike T-dualities [16] have been considered. It has been shown that dualities involving the timelike direction can change the signature of space-time [16] and lead to

[7] Here we follow the usual convention. The last label on the right refers to the fundamental weight associated with the 'time' root.

'exotic' phases of M-theory with more than one time. Starting with the 'orthodox' M-theory corresponding to 11-dimensional supergravity with signature $(T, S) = (1, 10)$ it has been shown [16] that one can reach by U-duality M^*-theory with $(T, S) = (2, 9)$ and the 'wrong' sign in front of the kinetic term of the four-form field strength F_4. One can also reach M'-theory with $(T, S) = (5, 6)$ and the conventional sign in front of the kinetic term of F_4. In the E_8^{+++} theory (and more generally in $\mathcal{G}^{+++}$) the usual signature of space-time is implemented through the temporal involution [10] which ensures that in the A_{10} (A_{D-1}) level decomposition we have SO(1,10) (SO$(1, D-1)$) tensors. The expected existence of the exotic phases in the E_8^{+++} context has been shown in ref. [14]. The author studied the Weyl reflections and showed that starting with the temporal involution one can reached the above-mentioned exotic phases using Weyl reflections involving the time direction. In other words, in contrast with the usual Cartan involution, the temporal involution is not invariant under conjugation by all the Weyl reflections. As a consequence, the E_8^{+++}-invariant theory proposed in [10] contains also the algebraic counterpart of single extremal brane solutions of the exotic phases M^* and M'.

The argument can be generalised to all $\mathcal{G}^{+++}$ theories. Indeed, a subset of the Weyl reflections of $\mathcal{G}^{+++}$ maps extremal branes onto other extremal branes generalising to all 'M-theories' the notion of dualities [10]. As pointed out in [14], we can expect that some Weyl reflections (around roots not on the gravity line and involving time direction) will not leave the temporal involution invariant and will lead to 'exotic' $\mathcal{G}^{+++}$ theories. These would correspond to maximally oxidised $\mathcal{G}$ with some (T, S) signatures and with possibly wrong sign kinetics terms for some form field strengths.

Even without having classified explicitly[8] the possible exotic phases for all $\mathcal{G}^{+++}$, we want to argue here that when such phases do exist the analysis of the previous section extend to them. We can first study the extremal branes of these maximally oxidised $\mathcal{G}$ theory with some fixed signature (T, S). Extremal branes of the exotic phases of M-theory (E_8) have been considered in [24]. The existence of intersecting brane solutions for a generic theory in $D = T + S$ dimensions with T timelike dimensions and S spacelike dimensions which includes gravity, a dilaton, form field strengths F_{n_I} of arbitrary degree n_I with arbitrary couplings to the dilaton a_I and 'n_I-form' kinetic terms with an arbitrary sign given by $\Theta_I = \pm 1$ ($\Theta = +1$ corresponding to the conventional sign) has been studied in ref. [26]. Each single extremal q_A-brane is characterised by s_A spatial longitudinal directions and t_A temporal longitudinal directions. There is furthermore a condition which has to be satisfied in order for the single brane solution to exist [24]:

$$\Theta_A(-1)^{t_A+1} = 1\,, \tag{3.1}$$

where $\Theta_A = \Theta_{I=q_A+2}$ associated with the F_I in the action when the q_A-brane is an electric one and $\Theta_A = (-1)^{T+1}\Theta_{I=D-q_A-2}$ when the q_A-brane is a magnetic one. The condition Eq,(3.1) is trivially satisfied in the orthodox phases. The generalised intersection rules give then for each pair (A, B) of extremal branes the following conditions involving the number

[8] For very-recent results on the exotic phases of the $\mathcal{G}$ theories see [25, appendix A] .

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of common spacelike directions $\bar{s}$ and the number of common timelike directions $\bar{t}$ [26]

$$\bar{s}+\bar{t}=\frac{(s_A+t_A)(s_B+t_B)}{D-2}-\frac{1}{2}\varepsilon_A a_A \varepsilon_B a_B\,. \qquad (3.2)$$

We note that the generalised intersection rule eq. (3.2) depends only on the total dimensions s_A+t_A, s_B+t_B and $\bar{s}+\bar{t}$ irrespectively of the temporal or spatial nature of them. It does not depend on the signature nor on the sign of the kinetic terms of the forms. Consequently in $\mathcal{G}^{+++}$ language eq. (3.2) can still be translated into the orthogonality condition proved in theorem 1, independent of the involution. Only the existence of a solution for the building blocks, namely the single extremal branes, depends on the particular involution through eq. (3.1). Each single extremal q_A-brane is characterised in $\mathcal{G}^{+++}$ by only one non-zero field component $A_{\tau_1\ldots\tau_{t_A}\lambda_1\ldots\lambda_{s_A}}$ where τ_i are timelike directions and λ_i spacelike ones. This field is the parameter of a antisymmetric tensor step operator of low level $R^{\tau_1\ldots\tau_{t_A}\lambda_1\ldots\lambda_{s_A}}$ and corresponds to the real positive root $\alpha_{(\tau_1\ldots\tau_{t_A},\lambda_1,\ldots,\lambda_{s_A})}$. All the analysis of section 2 can thus be repeated in this framework. Under a Weyl reflection transforming an orthodox intersecting brane configuration into an exotic one, the invariance of the lapse constraint eq. (2.26) , and particularly the sign of the last term is ensured by the condition eq. (3.1).

Thus to each intersecting extremal brane configuration of an 'exotic' maximally oxidised $\mathcal{G}$ theory there exists an algebraic counterpart which is an exact solution of the $\mathcal{G}^{+++}$ theory.

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